

AN APPLICATION OF HOMOLOGICAL ALGEBRA TO THE HOMOTOPY CLASSIFICATION OF TWO DIMENSIONAL CW-COMPLEXES

BY

MICHEAL N. DYER

ABSTRACT. Let π be $Z_m \times Z_n$. In this paper the homotopy types of finite connected two dimensional CW-complexes with fundamental group π are shown to depend only on the Euler characteristic. The basic method is to study the structure of the group $\text{Ext}_{Z\pi}^1(I\pi^2, Z)$ as a principal $\text{End}(I\pi^2)$ -module.

1. In this paper π will denote the noncyclic group $Z_m \times Z_n$, which is the product of two finite cyclic groups Z_m and Z_n . Thus the $\text{gcd}(m, n) \neq 1$. For convenience, we will always assume that m divides n . This is no restriction.

Let $X_{\mathcal{P}}$ denote the two dimensional CW-complex modeled on the presentation $\mathcal{P} = \{x, y: x^m, y^n, [x, y]\}$ of π and let $\pi_2 = \pi_2 X_{\mathcal{P}}$. $X_{\mathcal{P}}$ is called the standard model and π_2 the standard module. The study of this π -module π_2 forms the basis of this paper.

For any $\theta \in \text{Aut } \pi$ and any π -module M , the module ${}_{\theta}M$ has action given by $g * m = \theta(g)m$ for any $m \in M$, $g \in \pi$. Two modules M, N are said to be θ -isomorphic iff there is an isomorphism $\alpha: M \rightarrow {}_{\theta}N$. The module π_2 splits as a short exact sequence $Z \rightarrow \pi_2 \rightarrow (I\pi)^2$ where Z is the trivial π -module and $I\pi$ is the augmentation ideal in $Z\pi$. By studying the group $\text{Ext}(I\pi^2, Z)$ we prove the following crucial theorem.

THEOREM A. For any π -module M such that $M \oplus Z\pi \cong \pi_2 \oplus Z\pi$, we have $M \cong {}_{\theta}\pi_2$ for some $\theta \in \text{Aut } \pi$.

Hence, M is stably isomorphic to π_2 iff M is θ -isomorphic to π_2 for some $\theta \in \text{Aut } \pi$.

The group $H^3(\pi; \pi_2)$ is isomorphic to the cyclic group Z_{mn} [D₁, §2]; to each integer q prime to mn , there is a projective ideal $(q, N) \subset Z\pi$ generated by q and $N = (\sum_{i=1}^m x^i)(\sum_{j=1}^n y^j)$. The function $\partial: Z_{mn}^* \rightarrow \tilde{K}_0 Z\pi$ given by $\partial(q + (mn)) = \{(q, N)\} \in \tilde{K}_0 Z\pi$ is a homomorphism. A θ -isomorphism $\alpha: \pi_2 \rightarrow {}_{\theta}\pi_2$ has degree $k \in Z_{mn}^*$ iff $(\theta^*)^{-1}\alpha_*(1) = k$ in the diagram:

$$H^3(\pi; \pi_2) \xrightarrow{\alpha_*} H^3(\pi; {}_{\theta}\pi_2) \xleftarrow{\theta^*} H^3(\pi; \pi_2).$$

THEOREM B. For any $k \in \ker \partial \subset Z_{mn}^*$ there is a $\theta \in \text{Aut } \pi$ and a θ -isomorphism $\alpha: \pi_2 \rightarrow {}_{\theta}\pi_2$ of degree k .

Received by the editors April 17, 1979.

1980 *Mathematics Subject Classification*. Primary 57M20, 55U99.

© 1980 American Mathematical Society
0002-9947/80/0000-0260/\$03.50

We prove Theorem A in §4 and Theorem B in §5.

The following Corollaries 1 and 2 follow from A and B just as in [D₂, Theorem 5.5].

DEFINITION. A $(G, 2)$ -complex is a finite, connected, 2-dimensional CW-complex having fundamental group isomorphic to G .

COROLLARY 1. Any two $(Z_m \times Z_n, 2)$ -complexes have the same homotopy type iff they have the same Euler characteristic.

In the language of [D₂], the homotopy trees $\text{HT}(Z_m \times Z_n, 2)$ have essential height zero.

COROLLARY 2. Let X be a CW-complex with fundamental group isomorphic to $Z_m \times Z_n$ and suppose that X is dominated by a $(G, 2)$ -complex. Then X has the homotopy type of a $(Z_m \times Z_n, 2)$ -complex iff the Wall obstruction vanishes.

In the homotopy classification of G -complexes for G finite abelian, these results fill in a gap that existed between G cyclic [D₁] and G having more than two torsion coefficients [SD]. A technique similar to this may be decisive in determining the isomorphism and θ -isomorphism classes of the minimal $(G, 2)$ -modules detected in [SD], for G finite abelian.

2. A study of $\text{Ext}((I\pi)^2, Z)$. By looking at the cellular chain complex of the universal cover $\tilde{X}_\varphi$ of the standard model X_φ , we may identify π_2 as the kernel of the following exact sequence:

$$\mathcal{C}_*(\tilde{X}_\varphi) : \pi_2 \twoheadrightarrow (Z\pi)^3 \xrightarrow[\partial_2]{\begin{bmatrix} N_x & 1-y & 0 \\ 0 & x-1 & N_y \\ & \parallel & \end{bmatrix}} (Z\pi)^2 \xrightarrow[\partial_1]{(x-1, y-1)} Z\pi \xrightarrow{\epsilon} Z. \quad (2.1)$$

For an integer $r > 0$ and $z \in \pi$, let $\langle z, r \rangle = 1 + z + \cdots + z^{r-1}$. Then $N_x = \langle x, m \rangle$ and $N_y = \langle y, n \rangle$. The map $\epsilon: Z\pi \rightarrow Z$ is the augmentation homomorphism. It is easy to see that π_2 has generators the columns of the matrix

$$\begin{bmatrix} x-1 & y-1 & 0 & 0 \\ 0 & N_x & -N_y & 0 \\ 0 & 0 & x-1 & y-1 \end{bmatrix}.$$

Label the columns g_1, g_2, g_3 , and g_4 respectively.

Let $\eta_{13}: (Z\pi)^3 \rightarrow (Z\pi)^2$ denote the projection on the first and third coordinates. $\eta = \eta_{13}|_{\pi_2}$ has image $I\pi^2$ and kernel $\pi_2^\pi = \{\alpha \in \pi_2 \mid g\alpha = \alpha \text{ for all } g \in \pi\} = Z = Z\pi(0, N, 0)$, where $N = \langle x, m \rangle \langle y, n \rangle$. Thus the extension class $[\mathcal{E}]$ of the extension

$$\mathcal{E}: Z \twoheadrightarrow \pi_2 \xrightarrow{\eta} I\pi^2$$

is a member of $E = \text{Ext}_{Z\pi}^1(I\pi^2, Z)$. Sometimes, we will denote the class of the extension $\mathcal{F}: Z \twoheadrightarrow M \twoheadrightarrow I\pi^2$ by $[\mathcal{F}_M]$. Using the fact that $\text{Ext}_{Z\pi}^1(I\pi, Z) \cong H^2(\pi; Z) \cong \text{Ext}_Z(\pi, Z) \cong \pi$, we see that $\text{Ext}(I\pi^2, Z) \cong \pi^2$. We will think of E as

2×2 matrices

$$E = \begin{bmatrix} Z_n & Z_n \\ Z_m & Z_m \end{bmatrix}.$$

E may be considered as a right module over the ring $\text{End}(I\pi)^2$ as follows: to each $\alpha \in \text{End}(I\pi)^2$ and each extension class $[\mathcal{F}] \in E$ we associate the extension class $[\mathcal{F}\alpha]$ which is the pull-back of M by α . Thus

$$\begin{array}{ccccc} \mathcal{F}\alpha: & Z \twoheadrightarrow & M\alpha & \twoheadrightarrow & (I\pi)^2 \\ & \parallel & \downarrow & & \downarrow \alpha \\ \mathcal{F}: & Z \twoheadrightarrow & M & \twoheadrightarrow & (I\pi)^2 \end{array}$$

In fact, with this action, E becomes a *principal* $\text{End}(I\pi)^2$ -module with generator $[\mathcal{E}]$. To see this, we use the long exact sequence for $\text{Ext}_{Z\pi}^i$ associated with $\mathcal{E}$ [HS, p. 139]:

$$\text{Hom}(I\pi^2, \pi_2) \rightarrow \text{End}(I\pi)^2 \xrightarrow{\partial} \text{Ext}(I\pi^2, Z) \rightarrow \text{Ext}(I\pi^2, \pi_2) \rightarrow \dots$$

The boundary operator ∂ is described by $\partial(\alpha) = [\mathcal{E}\alpha]$. Sometimes, when the basic extension is clear, $\partial(\alpha)$ will be denoted by $[\alpha]$. But, by using the exact sequence 2.1, $\text{Ext}_{Z\pi}^1((I\pi)^2, \pi_2) \cong \text{Ext}_{Z\pi}^2(Z^2, \pi_2) \cong [H^2(\pi; \pi_2)]^2 = 0$ [D₁, Lemma 6.7]. Thus we have proved the following lemma.

2.2 LEMMA. $\text{Ext}((I\pi)^2, Z)$ is a principal $\text{End}(I\pi)^2$ -module with generator $[\mathcal{E}]$. $\square$

DEFINITION. Let $\mathcal{G}(E) = \{[\mathcal{F}] \in E \mid [\mathcal{F}] \text{ is a generator of } E \text{ as an } \text{End}(I\pi)^2\text{-module}\}$.

2.3 LEMMA. Suppose that M is stably isomorphic to the standard module π_2 ; i.e., $M \oplus Z\pi \cong \pi_2 \oplus Z\pi$. Then, if $M_\pi = M/M^\pi$, the extension $\mathcal{F}_M: Z = M^\pi \twoheadrightarrow M \twoheadrightarrow M_\pi \cong I\pi^2$ generates E as an $\text{End}((I\pi)^2)$ -module.

PROOF. If we can show that M is an extension of $Z = M^\pi$ by $I\pi^2$, then $[\mathcal{F}_M] \in \mathcal{G}(E)$ follows using the argument above (with the exact sequence $\mathcal{F}_M$) together with the fact that

$$H^2(\pi, M) \cong H^2(\pi; M \oplus Z\pi) \cong H^2(\pi; \pi_2 \oplus Z\pi) \cong H^2(\pi; \pi_2) = 0$$

since $H^2(\pi; Z\pi) = 0$ for any finite group [CE, p. 233]. To prove the first statement, observe that

$$M \oplus Z\pi \cong \pi_2 \oplus Z\pi \Rightarrow M_\pi \oplus Z\pi / (N) \cong (\pi_2)_\pi \oplus Z\pi / (N).$$

A careful, but elementary argument shows then that $M_\pi \oplus I\pi \cong (\pi_2)_\pi \oplus I\pi$. Because $I\pi$ (and hence $(I\pi)^2$) satisfies the Eichler condition [SE, p. 176] and $I\pi$ is a direct summand of $(\pi_2)_\pi \cong (I\pi)^2$, we have, using Jacobinski's cancellation theorem [SE, Theorem 19.8], that $M_\pi \cong I\pi^2$. $\square$

NOTE. For any $(\pi, 2)$ -complex Y , and any isomorphism $\alpha: \pi \rightarrow \pi_1 Y$ it follows that ${}_a\pi_2(Y) \oplus Z\pi \cong \pi_2 \oplus Z\pi$. These modules are therefore of topological interest. We will show in Theorem 4.2 that the *converse* is also true; that is,

$[\mathcal{F}] \in \mathcal{G}(E)$ implies that M is stably isomorphic to π_2 .

We identify the ring of endomorphisms of π ($= \text{End } \pi$) as a subset of E .

NOTATION. For each integer a , let $\bar{a}_k$ be the residue class of $a \pmod{k}$.

Let

$$\text{End } \pi = \left\{ \alpha = \begin{bmatrix} \bar{a}_n & \bar{b}_n \\ \bar{c}_m & \bar{d}_m \end{bmatrix} \in E = \begin{bmatrix} Z_n & Z_n \\ Z_m & Z_m \end{bmatrix} \mid n \text{ divides } bm \right\}.$$

Multiplication of two elements in E (as 2×2 -matrices) is well defined iff they are in $\text{End } \pi$. $\text{Aut } \pi \subset \mathcal{G}(E)$ is the subset of $\text{End } \pi \subset E$ consisting of invertible elements. Note that $\alpha(x^i y^j) = x^{cj+di} y^{aj+bi}$ can be computed from

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} j \\ i \end{pmatrix} = \begin{pmatrix} aj + bi \\ cj + di \end{pmatrix}$$

(observe that we have interchanged x and y).

In general $\mathcal{G}(E)$ is bigger than $\text{Aut } \pi$, as $\mathcal{G}(E)$ contains the image of $\text{GL}(2, Z)$ in E . For example,

$$\left[\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \right] = \begin{pmatrix} 1 & 1 \\ \bar{2}_m & 1 \end{pmatrix}$$

is always in $\mathcal{G}(E)$, but never in $\text{Aut } \pi$.

2.4 LEMMA. The boundary operator $\partial: \text{End}(I\pi)^2 \rightarrow E$ is described by carrying each

$$\alpha = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \quad (\alpha_{ij} \in Z\pi)$$

to

$$\left[\begin{array}{cc} \overline{\varepsilon(\alpha_{11})_n} & \overline{\varepsilon(\alpha_{12})_n} \\ \overline{\varepsilon(\alpha_{21})_m} & \overline{\varepsilon(\alpha_{22})_m} \end{array} \right].$$

PROOF. We are thinking of E as $\text{End}(I\pi)^2/B$, where

$$B = \left\{ \alpha \in \text{End}(I\pi)^2 \mid \alpha \text{ coextends to } \pi_2: \begin{array}{ccc} & \bar{\alpha} & (I\pi)^2 \\ & \searrow & \downarrow \alpha \\ \pi_2 & \xrightarrow{\eta} & (I\pi)^2 \end{array} \right\}.$$

B is always a right ideal, but it is not a left ideal unless $m = n$. The identification of E with π^2 is accomplished as follows: Identify each element α with a 2×2 matrix

$$\alpha = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$$

where each $\alpha_{ij} \in Z\pi$. This can be done because $\text{End } I\pi \cong Z\pi/(N)$ for any finite group π . By direct computation one may show that any map $\beta = (\beta_{ij})$ coextends, provided each $\beta_{ij} \in I\pi$. One simply shows directly that, if E^{ij} ($i, j = 1, 2$) denotes the elementary 2×2 matrix with a one in the ij th slot and zeros elsewhere, then $(x-1)E^{ij}$ and $(y-1)E^{ij}$ coextend. The β given above is a linear combination of $(x-1)E^{ij}$ and $(y-1)E^{ij}$ (because each $\beta_{ij} = \beta'_{ij}(x-1) + \beta''_{ij}(y-1)$), and hence

coextends. For example, $\beta = \begin{pmatrix} 0 & 0 \\ x-1 & 0 \end{pmatrix}$ coextends by the map $\bar{\beta}: (I\pi)^2 \rightarrow \pi_2$ given by defining $\bar{\beta}(x-1, 0) = (x-1)(0, -N_y, x-1)$, $\bar{\beta}(y-1, 0) = (y-1)(0, 0, x-1)$, and $\bar{\beta}(0, y-1) = 0 = \bar{\beta}(0, x-1)$. Then $\eta \circ \bar{\beta} = \beta$ and we are done *provided* $\bar{\beta}$ is well defined. Using 2.1, we identify $\text{Hom}_{Z\pi}(I\pi, M)$ with $\{\alpha: (Z\pi)^2 \rightarrow M: \alpha|_{\text{im } \partial_2} = 0\}$. It is easy to check that the map $\bar{\alpha}: (Z\pi)^2 \rightarrow \pi_2$ which sends $(1, 0) \rightarrow (0, -(x-1)N_y, (x-1)^2)$ and $(0, 1) \rightarrow (0, 0, (y-1)(x-1))$ is zero when restricted to $\text{im } \partial_2$. Then $\bar{\beta} = (\bar{\alpha}, 0): (I\pi)^2 \rightarrow \pi_2$.

Thus each $\alpha = (\alpha_{ij})$ in $\text{End}(I\pi)^2$ is equivalent mod B to the map $(\epsilon(\alpha_{ij}))$ with integer entries. One may further show that the matrices nE^{1j} and mE^{2j} coextend ($j = 1, 2$). For example, $\langle y, n \rangle E^{11}$ coextends via a map $(I\pi)^2 \rightarrow \pi_2$ defined by carrying $(x-1, 0) \mapsto ((x-1)\langle y, n \rangle, 0, 0)$ and $(y-1, 0), (0, x-1), (0, y-1)$ all to zero. Thus we see that the map $\partial: \text{End}(I\pi)^2 \rightarrow \text{Ext}((I\pi)^2, Z)$ can be described by

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \mapsto \begin{bmatrix} \frac{\epsilon(\alpha_{11})_n}{\epsilon(\alpha_{21})_m} & \frac{\epsilon(\alpha_{12})_n}{\epsilon(\alpha_{22})_m} \end{bmatrix}. \quad \square$$

We will isolate several subsets of $\mathcal{G}(E)$, which we then proceed to study. Let

$$S\mathcal{G}(E) = \{[\mathcal{F}_M] \in E | M \oplus Z\pi \cong \pi_2 \oplus Z\pi\},$$

$${}_{\theta}\text{Iso} = \{[\mathcal{F}_M] \in E | M \cong_{\theta} \pi_2 \text{ for some } \theta \in \text{Aut } \pi\}, \text{ and}$$

$$\text{Iso} = \{[\mathcal{F}_M] \in E | M \cong \pi_2\}.$$

Clearly, by 2.3, any module M stably isomorphic to π_2 is already an extension of Z by $I\pi^2$.

The following inclusions hold:

$$\mathcal{G}(E) \supset S\mathcal{G}(E) \supset {}_{\theta}\text{Iso} \supset \text{Iso}.$$

The first inclusion follows from Lemma 2.3. The last inclusion is clear; the second follows as any module M which is θ -isomorphic to π_2 may be embedded in the sequence: $0 \rightarrow M \rightarrow {}_{\theta}C_2 \rightarrow {}_{\theta}C_1 \rightarrow {}_{\theta}Z\pi \rightarrow Z$; thus M is stably isomorphic to π_2 by Schanuel's lemma.

One can show that $\mathcal{G}(E) = \{[\mathcal{F}_M] \in E | M \text{ has the same genus as } \pi_2\}$. However, it is not true that any module M of the same genus as π_2 is an extension of Z by $I\pi^2$.

For future reference, we record the following easily proved characterization of Iso .

2.5 PROPOSITION. *Let $[\mathcal{F}_M] \in \mathcal{G}(E)$. Then $M \cong \pi_2$ iff $[\mathcal{F}] = [\mathcal{E}\alpha] \in E$ for some $\alpha \in \text{GL}(2, Z\pi/(N)) \subset \text{End}(I\pi)^2$. $\square$*

3. Comparison of ${}_{\theta}\pi_2$ and $\pi_2\theta$. In this section we compare the pullback $\pi_2\theta$ and the module ${}_{\theta}\pi_2$ for any $\theta \in \text{Aut } \pi$. We continue the assumption that $m|n$.

Let ${}_{\theta}\mathcal{E}$ denote the extension $Z \rightarrow {}_{\theta}\pi_2 \rightarrow {}_{\theta}I\pi^2 \approx I\pi^2$, with the isomorphism given by $\begin{bmatrix} Z\theta^{-1} & 0 \\ 0 & Z\theta^{-1} \end{bmatrix}$. Recall that any $\theta \in \text{Aut } \pi$ can be written as a product $\theta = E_1 E_2 \dots E_k D$ where each $E_i = \begin{bmatrix} 1 & a_i \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ b_i & 1 \end{bmatrix}$ ($a_i, b_i \in Z$ and n divides ma_i , $i = 1, \dots, k$) is an elementary automorphism and $D = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$ ($p, q \in Z$) is a diagonal automorphism, provided p is prime to n and q is prime to m [S, Proposition 6].

DEFINITION. If $\theta = \begin{bmatrix} 1 & -r \\ 0 & 1 \end{bmatrix}$ (respectively, $\begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix}$, $\begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$) is a member of $\text{Aut } \pi$ (i.e., mr is divisible by n) then let

$$\theta^0 = \begin{bmatrix} 1 & 0 \\ rm/n & 1 \end{bmatrix} \quad \left(\text{respectively } \begin{bmatrix} 1 & sn/m \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} \right).$$

Then, for any $\theta \in \text{Aut } \pi$, θ^0 is the element of $\text{Aut } \pi$ defined by writing $\theta = E_1 \dots E_k \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$, where $E_1, \dots, E_k$ are elementary automorphisms, and setting $\theta^0 = \theta_1^0 \dots \theta_k^0 \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$. Notice that $(\theta^0)^0 = \theta$.

3.1 THEOREM. For any $\theta \in \text{Aut } \pi$, $[\theta \mathcal{E}] = [\mathcal{E} \theta^0]$ and $[\mathcal{E} \theta] = [\theta^0 \mathcal{E}]$.

PROOF. Let $\theta \in \text{Aut } \pi$ be written as a product of a diagonal automorphism $D = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$ and a product $E_1 \dots E_k$ of elementary automorphisms. The theorem will follow from the fact that $[\mathcal{E}(\alpha\beta)] = [(\mathcal{E}\alpha)\beta]$ and ${}_{(\alpha\beta)}\pi_2 = {}_{\beta}({}_{\alpha}\pi_2)$ provided we can show that the theorem is true for D and E_i . We will give only the proof for $\theta = \begin{bmatrix} 1 & 0 \\ -r & 1 \end{bmatrix}$ (i.e. $\theta(x) = x$, $\theta(y) = x^{-r}y$), as the others are similar.

3.2 LEMMA. Let $\pi = Z_m \times Z_n$, with $m|n$, and $\theta = \begin{bmatrix} 1 & 0 \\ -r & 1 \end{bmatrix}$, $\theta^0 = \begin{bmatrix} 1 & m/m \\ 0 & 1 \end{bmatrix}$. Then $[\theta \mathcal{E}] = [\mathcal{E} \theta^0]$ in $\text{Ext}(I\pi^2, Z)$.

PROOF. Let $\beta = \sum_{i=1}^n y^{i-1} \langle x, ri \rangle$, $k = rn/m$, and $\bar{\theta} = \theta^{-1} = \begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix}$. Straightforward calculation using 2.1 shows that the matrix

$$\begin{bmatrix} 1 & 0 & k \\ 0 & x^r & \beta \\ 0 & 0 & 1 \end{bmatrix}$$

defines a map from $C_2 = Z\pi e_1 \oplus Z\pi e_2 \oplus Z\pi e_3 \rightarrow {}_{\theta}C_2 = {}_{\theta}Z\pi e_1 \oplus {}_{\theta}Z\pi e_2 \oplus {}_{\theta}Z\pi e_3$ (i.e., $e_1 \rightarrow (1, 0, 0)$, $e_2 \rightarrow (0, x^r, 0)$, $e_3 \rightarrow (k, \beta, 1)$ and extend linearly) which sends π_2 ($\subset C_2$) into ${}_{\theta}\pi_2$ ($\subset {}_{\theta}C_2$). This same matrix defines a map from ${}_{\theta}C_2 \rightarrow {}_{\theta}({}_{\theta}C_2) = C_2$ and hence a map $\theta_2: {}_{\theta}\pi_2 \rightarrow \pi_2$. On the generators for ${}_{\theta}\pi_2$, θ_2 looks like

$$\begin{aligned} \bar{\theta}(x-1) * {}_{\theta}e_1 &= (x-1)e_1 \\ &\mapsto \bar{\theta}(x-1)e_1, \bar{\theta}(y-1) * {}_{\theta}e_1 + \bar{\theta}(N_x) * {}_{\theta}e_2 = (y-1, N_x, 0) \\ &\mapsto \bar{\theta}(y-1)e_1 + x^r \bar{\theta}(N_x)e_2, \bar{\theta}(-N_y) * {}_{\theta}e_2 + \bar{\theta}(x-1) * {}_{\theta}e_3 \\ &= (0, -N_y, x-1) \\ &\mapsto k \bar{\theta}(x-1)e_1 + (-x^r \bar{\theta}(N_y) + \bar{\theta}(x-1)\beta)e_2 + \bar{\theta}(x-1)e_3 \end{aligned}$$

and finally,

$$\bar{\theta}(y-1) * {}_{\theta}e_3 = (y-1)e_3 \mapsto k \bar{\theta}(y-1)e_1 + \beta \bar{\theta}(y-1)e_2 + \bar{\theta}(y-1)e_3.$$

Inspection shows that θ_2 induces a map $\theta'_2: {}_{\theta}(I\pi)^2 \rightarrow (I\pi)^2$ with matrix $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$. Here, $1: {}_{\theta}I\pi \rightarrow I\pi$ is the map induced by $Z\bar{\theta}: {}_{\theta}Z\pi \rightarrow Z\pi$. Let $Z\theta$ denote the isomorphism $I\pi \rightarrow {}_{\theta}I\pi$ induced from $Z\theta: Z\pi \rightarrow {}_{\theta}Z\pi$. The composition

$$(I\pi)^2 \xrightarrow{\begin{pmatrix} Z\theta & 0 \\ 0 & Z\theta \end{pmatrix}} {}_{\theta}(I\pi)^2 \xrightarrow{\theta'_2} (I\pi)^2$$

yields a map with matrix $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$. $\square$

4. Characterizing $\mathcal{G}(E)$. We will now characterize the set $\mathcal{G}(E)$ of generators of $E = \text{Ext}(I\pi^2, Z)$. For each $\alpha = (\alpha_{ij}) \in \text{End}(I\pi)^2$ ($\alpha_{ij} \in Z\pi$), let $\varepsilon(\alpha)$ denote the integer matrix with entries $\varepsilon(\alpha_{ij})$ ($i, j = 1, 2$).

4.1 PROPOSITION. *Let $\alpha = (\alpha_{ij}) \in \text{End}(I\pi)^2$ ($\alpha_{ij} \in Z\pi$). The following are equivalent:*

- (a) $[\alpha] \in \mathcal{G}(E)$.
- (b) $\exists \alpha' \in \text{End}(I\pi)^2$ such that $\alpha\alpha' - 1 \in B$.
- (c) The determinant of $\varepsilon(\alpha)$ is prime to m and there are integers s, t such that $\varepsilon(\alpha_{11})s + \varepsilon(\alpha_{12})t \equiv 1 \pmod{n}$.
- (d) There is an integer matrix $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that

$$\bar{\gamma}_n = \begin{bmatrix} \bar{a}_n & \bar{b}_n \\ \bar{c}_n & \bar{d}_n \end{bmatrix} \in \text{GL}(2, Z_n)$$

and $[\alpha] = [\gamma] \in E$.

PROOF. (a) $\Rightarrow$ (b). If $[\mathcal{E}\alpha] \in \mathcal{G}(E)$, then $[\mathcal{E}]$ itself must be a pullback of $Z \rightarrow \pi_2\alpha \rightarrow I\pi^2$ by some $\alpha': I\pi^2 \rightarrow I\pi^2$. Thus $[\mathcal{E}] = [\mathcal{E}\alpha\alpha']$ implies $\alpha\alpha' - 1 \in B$.

(b) $\Rightarrow$ (c). It is easy to see that the correspondence $[\alpha] \in E \mapsto \det \varepsilon(\alpha) \in Z$ is well defined modulo m . $\alpha\alpha' - 1 \in B$ implies $\det \varepsilon(\alpha) \cdot \det \varepsilon(\alpha') \equiv 1 \pmod{m}$. Thus $\det \varepsilon(\alpha)$ is prime to m . Furthermore, $\varepsilon(\alpha_{11})\varepsilon(\alpha'_{11}) + \varepsilon(\alpha_{12})\varepsilon(\alpha'_{21}) - 1 \equiv 0 \pmod{n}$ follows by looking at the 11-coordinate of $\partial(\alpha\alpha' - 1)$.

(c) $\Rightarrow$ (d). Because m divides n , the natural map $\bar{a}_n \mapsto \bar{a}_m$ ($a \in Z$) induces a surjection $Z_n^* \rightarrow Z_m^*$. Let $d = \det \varepsilon(\alpha)$. d is prime to m , so there is an integer l such that $d + lm$ is prime to n . $s\varepsilon(\alpha_{11}) + t\varepsilon(\alpha_{12}) = 1 + kn$ then yields $d + l(1 + kn)m$ is prime to n . Now consider the integer matrix

$$\gamma = \begin{bmatrix} \varepsilon(\alpha_{11}) & \varepsilon(\alpha_{12}) \\ \varepsilon(\alpha_{21}) - ltm & \varepsilon(\alpha_{22}) + lsm \end{bmatrix}.$$

We claim that γ represents an element of $\text{GL}(2, Z_n)$; i.e., that $\det \gamma$ is prime to n . Consider

$$\det \gamma = d + lm(\varepsilon(\alpha_{11}) \cdot s + \varepsilon(\alpha_{12}) \cdot t) = d + lm(1 + kn),$$

which is prime to n . Clearly $[\gamma] = [\alpha]$, by Lemma 2.4.

(d) $\Rightarrow$ (a). Choose an integer matrix γ so that $[\gamma] = [\alpha]$ in E and $\bar{\gamma}_n \in \text{GL}(2, Z_n)$. Let γ' be an integer matrix which represents the inverse $(\bar{\gamma}_n)^{-1}$. Then $(\gamma\gamma' - 1)_n = 0$ implies that $\partial(\gamma\gamma' - 1) = 0$. We claim that $\alpha\gamma' - 1 \in B$. Choose $b \in B$ so that $\alpha = \gamma + b$. Then $\alpha\gamma' - 1 = (\gamma + b)\gamma' - 1 = \gamma\gamma' - 1 + b\gamma'$. But B is a right ideal, so $b\gamma' \in B$. Hence $\alpha\gamma' - 1 \equiv \gamma\gamma' - 1 \equiv 0 \pmod{B}$ and $\alpha \in \mathcal{G}(E)$. $\square$

4.2 THEOREM. $\mathcal{G}(E) = \theta \text{Iso}$.

PROOF. By Proposition 4.1, given any $[\alpha] \in \mathcal{G}(E)$ there is a 2×2 integer matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ whose determinant is prime to n , and such that $[A] = [\alpha] \in E$. Consider $\bar{A}_n$ as the matrix mod n , that is to say, A represents an element of $\text{GL}(2, Z_n) = \text{Aut}(Z_n \times Z_n)$. By Proposition 6 of [S], $A \equiv E_1 E_2 \dots E_k D \pmod{n}$ where $D = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$ is a diagonal automorphism (p, q are prime to n) and each E_i is an elementary

matrix of the form $\begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ \gamma & 1 \end{bmatrix}$, with $p, q, \gamma_i \in Z$. Let $M = E_1 E_2 \cdots E_k D$ be the integer matrix which is the product of $E_1, E_2, \dots, E_k$, and D . Then

$$\begin{aligned} [\mathcal{E}\alpha] &= [\mathcal{E}A] = [\mathcal{E}M] \quad (\text{because } \alpha \equiv A \equiv M \pmod{B}) \\ &= [(\dots ((\mathcal{E}E_1)E_2) \cdots E_k)D] \end{aligned}$$

(because pullbacks commute with composition).

But, by Proposition 2.5

$$\pi_2 \cong \pi_2 E_1 \cong \pi_2 E_1 E_2 \cong \cdots \cong \pi_2 E_1 \cdots E_k$$

because each $E_i \in \text{Aut}(I\pi)^2$. D is a member of $\text{Aut } \pi$ implies that

$$\pi_2 \cong \pi_2 E_1 \cdots E_k \cong {}_D \pi_2 M \cong {}_D \pi_2 \alpha$$

by invoking Theorem 3.1. $\square$

We have seen in §2 that the map $\bar{\partial} = \partial|_{\text{GL}(2, Z\pi/(N))}: \text{GL}(2, Z\pi/(N)) \rightarrow \mathcal{G}(E)$ is onto the subset Iso. We would like to determine $\text{im } \bar{\partial}$. As above, each elementary automorphism $\alpha = \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ \gamma & 1 \end{bmatrix}$ yields $\pi_2 \alpha \cong \pi_2$. The diagonal automorphisms $\beta = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$ ($p, q \in Z$) of π which are clearly in $\text{im } \bar{\partial}$ have $\bar{p}_{mn}, \bar{q}_{mn} \in \ker\{\partial: Z_{mn}^* \rightarrow \tilde{K}_0 Z\pi\}$; for, choosing units mod N $u, v \in Z\pi/(N)^*$ such that $\varepsilon(u) = p$, $\varepsilon(v) = q$ [4, 2.1], we have $\alpha = \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} \in \text{GL}(2, Z\pi/(N))$ and $\bar{\partial}\alpha = \beta$. We make the following convenient hypothesis [SD, §4].

REDUCTION HYPOTHESIS. The natural projection $Z_{mn}^* \rightarrow Z_n^*$ remains surjective when restricted to $\ker \partial$.

The reduction hypothesis is satisfied whenever $n = m = p$, an odd prime [U]. Whenever the reduction hypothesis is satisfied, the above argument implies that $\bar{\partial}$ is surjective.

4.3 COROLLARY. *If the reduction hypothesis is true, then each $[\alpha] \in \mathcal{G}(E)$ has $\pi_2 \alpha \cong \pi_2$; i.e., $\mathcal{G}(E) = \text{Iso}$.* $\square$

5. Shuffling k -invariants. Recall that $\pi = Z_m \times Z_n$, with $m|n$ and generators x, y of order m and n , respectively. For each π -module M , $H^3(\pi; M)$ is isomorphic to $\text{Hom}_{Z\pi}(\pi_2, M)/\mathcal{B}$, where $\mathcal{B} = \{\beta: \pi_2 \rightarrow M | \beta \text{ extends to a map } \tilde{\beta}: C_2 \rightarrow M\}$. For each $\alpha \in \text{Hom}_{Z\pi}(\pi_2, M)$, let $\{\alpha\} = \alpha + \mathcal{B}$ be the class of α in $H^3(\pi; M)$. If θ is a member of $\text{Aut } \pi$, then $\theta^*: H^*(\pi; \pi_2) \rightarrow H^*(\pi; {}_\theta \pi_2)$ may be computed by choosing a chain map from $\mathcal{C}_*(\tilde{X}_\varphi) \rightarrow {}_\theta \mathcal{C}_*(\tilde{X}_\varphi)$ covering the identity.

$$\begin{array}{ccccccc} & & C_2 & & & & \\ & & \parallel & & & & \\ \pi_2 & \rightarrow & (Z\pi)^3 & \xrightarrow{\partial_2} & Z\pi^2 & \xrightarrow{\partial_1} & Z\pi \rightarrow Z \\ \downarrow \theta_2 & & \downarrow \psi & & \downarrow & & \downarrow Z\theta \quad \parallel \\ {}_\theta \pi_2 & \rightarrow & {}_\theta (Z\pi)^3 & \xrightarrow{\partial_2} & {}_\theta Z\pi^2 & \xrightarrow{\partial_1} & {}_\theta Z\pi \rightarrow Z \end{array}$$

It follows that for any $\alpha \in \text{End } \pi_2$, $\theta^*\{\alpha\} = \{\theta_2 \alpha\} \in H^3(\pi; {}_\theta \pi_2)$.

If $\theta = \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}$, where p is prime to m (i.e., $\theta(x) = x^p$, $\theta(y) = y$), then

$$\psi = \begin{bmatrix} \langle x, p \rangle & 0 & 0 \\ 0 & \langle x, p \rangle & 0 \\ 0 & 0 & 1 \end{bmatrix}: (Z\pi)^3 \rightarrow {}_{\theta}(Z\pi)^3 \text{ induces } \theta_2: \pi_2 \rightarrow {}_{\theta}\pi_2.$$

The following lemma will prove useful.

5.1 LEMMA. *Let $v \in Z\pi$ be a unit (mod N) having augmentation $\epsilon(v) = p$. Then there is a unit (mod N) $u \in Z\pi$ having $\epsilon(u) = p$ and such that $u = \langle x, p \rangle + (y - 1)\alpha$, for some $\alpha \in Z\pi$.*

PROOF. For each $v \in Z\pi$, let $v_y \in Z(Z_m(x))$ denote the image of v under the map $\epsilon_y: Z\pi \rightarrow Z(Z_m)$ obtained by setting $y = 1$. v is a unit mod N implies v_y is a unit mod N_x , which in turn implies that $v_y = w\langle x, p \rangle$, where w is a unit in $Z(Z_m)$. w is also a unit in $Z\pi$ so $u = w^{-1}v$ has $u_y = \langle x, p \rangle$. Hence $u - \langle x, p \rangle$ is in the kernel of the map ϵ_y . It is easy to see that $u - \langle x, p \rangle = (y - 1)\alpha$ for some $\alpha \in Z\pi$. $\square$

Recall the homomorphism $\partial: Z_{mn}^*$ ($=$ units in $H^3(\pi; \pi_2)$) $\rightarrow \tilde{K}_0 Z\pi$ given by $\partial(\bar{p}_{mn}) = \{(p, N)\} \in \tilde{K}_0 Z\pi$. It is known that $\bar{p}_{mn} \in \ker \partial$ iff there is a $v \in Z\pi$ which is a unit mod N whose augmentation $\epsilon(v) = p$, [D₂, 2.1]. A θ -homomorphism $\alpha: \pi_2 \rightarrow {}_{\theta}\pi_2$ has degree $k \in H^3(\pi; \pi_2)$ iff $k\{\theta_2\} = \{\alpha\}$. For each $k \in \ker \partial$, we will construct a θ -isomorphism $\pi_2 \rightarrow {}_{\theta}\pi_2$ of degree k . This will prove Theorem B. We commence the proof.

Given $\bar{k}_{mn} \in \ker \partial \subset Z_{mn}^*$. Choose $u \in Z\pi$ such that $\epsilon(u) = k$ and u is a unit mod N . k is prime to mn implies that k is prime to m^2n . Choose an integer p such that $pk + sm^2n = 1$. Let θ be the automorphism which carries $x \rightarrow x^p$, $y \rightarrow y$. $\bar{p}_{mn} = \bar{k}_{mn}^{-1}$ in Z_{mn}^* implies $\bar{p}_{mn} \in \ker \partial$. Thus there is a $v \in Z\pi$ such that $\epsilon(v) = p$, v is a unit mod N , and $v = \langle x, p \rangle + (y - 1)\alpha$ for some $\alpha \in Z\pi$.

Let e_i ($i = 1, 2, 3$) denote the natural basis for $(Z\pi)^3$. Define homomorphisms $p_{ij}: (Z\pi)^3 \rightarrow {}_{\theta}(Z\pi)^3$ sending $e_i \rightarrow g_j$ (g_j is a generator of π_2 , hence, of ${}_{\theta}\pi_2$, see §2) and $e_k \rightarrow 0$ ($k \neq i$) ($i = 1, 2, 3$; $j = 1, 2, 3, 4$). Note that $\bar{p}_{ij} = p_{ij}|_{\pi_2}$ defines a degree 0 map from $\pi_2 \rightarrow {}_{\theta}\pi_2$.

Consider the map $M = \psi + \alpha p_{12}: (Z\pi)^3 \rightarrow {}_{\theta}(Z\pi)^3$. M has matrix

$$\begin{bmatrix} v = \langle x, p \rangle + (y - 1)\alpha & 0 & 0 \\ \alpha N_x & \langle x, p \rangle & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\bar{M} = M|_{\pi_2}: \pi_2 \rightarrow {}_{\theta}\pi_2$ carries $g_1 \mapsto (x^p - 1)(v, 0, 0)$, $g_2 \mapsto (v(y - 1), \alpha N_x(y - 1) + pN_x, 0)$, $g_3 \mapsto (0, -N_y\langle x, p \rangle, x^p - 1)$, and $g_4 \mapsto (0, 0, y - 1)$. The map $\bar{M}: (I\pi)^2 \rightarrow {}_{\theta}(I\pi)^2$ induced by $\bar{M}$ then has matrix $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and is an isomorphism because v is a unit (mod N).

We will now alter $\bar{M}$ to give an isomorphism $\pi_2 \rightarrow {}_{\theta}\pi_2$ of degree $\bar{k}_{mn}$. Look at $Q = uM + sNp_{22}: (Z\pi)^3 \rightarrow {}_{\theta}(Z\pi)^3$. Q then has matrix

$$\begin{bmatrix} uv & 0 & 0 \\ \alpha u N_x & u\langle x, p \rangle + smN & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This clearly restricts to a map $\bar{Q}: \pi_2 \rightarrow_{\theta} \pi_2$ having degree $\bar{k}_{mn}$ because $\{\bar{Q}\} = \{u\theta_2 + u\alpha\bar{p}_{12} + sN\bar{p}_{22}\} = \{u\theta_2\} = \{\varepsilon(u)\theta_2\} = k\{\theta_2\}$. Clearly $\bar{Q}$ induces the same map as $u\bar{M}$ on $I\pi^2 \rightarrow_{\theta} I\pi^2$ ($sN\bar{p}_{22}$ restricted to $(I\pi)^2$ is zero) and is an isomorphism. We need only show that $Q|_{\pi_2^*}$ is the identity: $Q(0, N, 0) = (u\langle x, p \rangle + smN)(0, N, 0) = (kp + sm^2n)(0, N, 0) = (0, N, 0)$. $\square$

REFERENCES

- [CE] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, N. J., 1956.
- [D₁] M. Dyer, *Homotopy classification of (π, m) -complexes*, J. Pure Appl. Algebra 7 (1976), 429–482.
- [D₂] ———, *On the essential height of homotopy trees with finite fundamental group*, Compositio Math. 36 (1978), 209–224.
- [HS] P. Hilton and U. Stammach, *A course in homological algebra*, Springer-Verlag, New York, 1970.
- [S] A. Sieradski, *Combinatorial isomorphisms and combinatorial homotopy equivalences*, J. Pure Appl. Algebra 7 (1976), 59–95.
- [SD] A. Sieradski and M. Dyer, *Distinguishing arithmetic for certain stably isomorphic modules*, J. Pure Appl. Algebra 15 (1979), 199–217.
- [SE] R. G. Swan and E. G. Evans, *K-theory of finite groups and orders*, Lecture Notes in Math., vol. 149, Springer-Verlag, Berlin and New York, 1970.
- [U] S. Ullom, *Non-trivial lower bounds for class groups of integral group rings*, Illinois J. Math. 20 (1976), 314–329.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OREGON 97403